Lecture 15 :

Infinite State Spaces

Def 1. The matrix $L := P - I$ is called Laplace Matrix.

For the exit distribution $h(x) = \mathbb{P}_x(V_A < V_B)$, it satisfies the Laplace Equation :

$$\begin{cases} Lh = 0, & \text{on } C = \mathcal{X} \setminus (A \cup B) ; \\ h = 1, & \text{on } A ; \\ h = 0, & \text{on } B . \end{cases}$$

Boundary Conditions

For the exit time $g(x) = \mathbb{E}_x T$, it solves the Poisson Equation :

$$\begin{cases} Lg = -1 & \text{on } C ; \\ g = 0 & \text{on } \mathcal{X} \setminus C . \end{cases}$$

— Boundary Conditions

Def 2. A state $x \in \mathcal{X}$ is said to be positive recurrent if $\mathbb{E}_x \tau_x < \infty$; is said to be null recurrent if it is recurrent but not positive recurrent, i.e., $P_x(\tau_x < \infty) = 1$ but $\mathbb{E}_x \tau_x = \infty$.

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Ex 1. (Reflecting Random Walk on $\mathbb{Z}$). Let $\mathcal{X} = \{0, 1, 2, \cdots\}$.

The transition probability is

$$\mathbb{P}(X_1 = y \mid X_0 = x) = \begin{cases} p, & y = x+1 ; \\ 1-p, & y = x-1 \ (x > 0) ; \\ 1-p, & y = x = 0 ; \\ 0, & \text{else} . \end{cases}$$

Here $0 < p < 1$. Thus, this chain is irreducible.

Case I. $0 < p < \frac{1}{2}$.

Notice

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{bmatrix} 1-p & p & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & \cdots \\ 0 & 1-p & 0 & p & \cdots \\ 0 & 0 & 1-p & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Solving

$$\begin{cases} \vec{\pi} P = \vec{\pi} & \Rightarrow \left(\vec{\pi}_{i+1} = \frac{p}{1-p} \vec{\pi}_i \right) . \\ \vec{\pi}_i \geq 0 , & \forall i \in \mathbb{N} \\ \sum_{i=0}^{\infty} \vec{\pi}_i = 1 \end{cases}$$

gives

$$\vec{\pi}_i = \frac{1-2p}{1-p} \cdot \left(\frac{p}{1-p} \right)^i > 0 , \quad \forall i \in \mathbb{N}.$$

Since $P_{00} > 0$, the state 0 has period 1.

Thus, 0 is aperiodic. Because this chain is irreducible, it is aperiodic.

Lemma 11.1 implies that the chain is recurrent.

i.e., $\rho_{ii} = \mathbb{P}_i(\tau_i < \infty) = 1, \quad \forall i \in \mathbb{N}$.

Theorem 9.3 implies that

$$\mathbb{E}_i \tau_i = \frac{1}{\pi_i} < \infty .$$

Case II. $\frac{1}{2} < p < 1$.

Let $h(x,N) := \mathbb{P}_x(\tau_0 < \tau_N)$, then it satisfies

$$(*) \quad \begin{cases} h(x,N) = (1-p)\,h(x-1,N) + p\,h(x+1,N), \ \forall\ 1 < x < N-1; \\ h(1,N) = (1-p) + p\,h(2,N); \\ h(N-1,N) = (1-p)\,h(N-2,N). \end{cases}$$

Solving this linear system of equations, one has

$$h(x,N) = 1 - \frac{1-\theta^x}{1-\theta^N}, \quad \forall \quad 1 \le x \le N-1.$$

Here $\theta = \frac{1-p}{p} < 1$.

Taking $N \longrightarrow \infty$ at both sides yields

$$\mathbb{P}_x(\tau_0 < \infty) = \theta^x < 1 \quad , \quad \forall\ x \geq 1.$$

In particular, $\mathbb{P}_1(\tau_0 < \infty) < 1$. Since 0 communicates with 1, we know 0 is transient. (Thm 1 in Lecture 5)

why?

And thus the chain is transient.

Case III. $p = \frac{1}{2}$.

Let $h(x,N) := \mathbb{P}_x(\tau_0 < \tau_N)$, then it satisfies (*).

Solving this linear system of equations, one has

$$h(x,N) = \frac{N-x}{N} \quad , \quad \forall \quad 1 \leq x \leq N-1.$$

Taking $N \longrightarrow \infty$ at both sides yields

$$\mathbb{P}_x(\tau_0 < \infty) = 1 \quad , \quad \forall\ x \geq 1.$$

Thus, $P_{00} = \mathbb{P}_0(\tau_0 < \infty) = \frac{1}{2} + \frac{1}{2} \cdot \mathbb{P}_1(\tau_0 < \infty) = 1$. This implies 0 is recurrent and so is the chain.

For any fixed $N > 3$, let $T_N := \min\{n \geq 1 : X_n \in \{0, N\}\}$.

For any $1 \leq x \leq N-1$, define $g(x,N) = \mathbb{E}_x T_N$. Then it solves

$$\begin{cases} g(x,N) = 1 + \frac{1}{2} g(x-1,N) + \frac{1}{2} g(x+1,N), & \forall\ 1 < x < N-1; \\ g(1,N) = 1 + \frac{1}{2} g(2,N); \\ g(N-1,N) = 1 + \frac{1}{2} g(N-2,N). \end{cases}$$

Solving this linear system of equations, one has

$$g(x,N) = x(N-x), \quad \forall\ 1 \leq x \leq N-1.$$

Notice that for any $1 \leq x \leq N-1$, $T_N \leq T_0$. Thus,

$$\forall\ 1 \leq x \leq N-1, \quad \mathbb{E}_x T_0 \geq \mathbb{E}_x T_N = g(x,N) = x(N-x).$$

In particular, $\mathbb{E}_1 T_0 \geq N-1$, for any $N > 3$.

Taking $N \longrightarrow \infty$ at both sides yields $\mathbb{E}_1 T_0 = \infty$.

Thus, $\mathbb{E}_0 T_0 = 1 + \frac{1}{2} \mathbb{E}_1 T_0 = \infty$.

In sum, if $p \in (\frac{1}{2}, 1)$, 0 is transient;

if $p = \frac{1}{2}$, 0 is null recurrent.

if $p \in (0, \frac{1}{2})$, 0 is positive recurrent.

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Theorem 15.1 For an irreducible chain, the following are equivalent:

(i). Some state is positive recurrent.

(ii). There is a stationary distribution.

(iii). All states are positive recurrent.

Proof. We will show (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

$\boxed{\vec{\mu}^x_y = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y,\ \tau_x > n)}$

(i)⇒(ii). Recall that (in the proof of Proposition 11.1) the stationary measure $\vec{\mu}^x$ has a total mass

$$\begin{aligned}\sum_{y\in\mathcal{X}} \vec{\mu}^x_y &= \sum_{y\in\mathcal{X}} \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y,\ \tau_x > n) \\ &= \sum_{n=0}^{\infty} \sum_{y\in\mathcal{X}} \mathbb{P}_x(X_n = y,\ \tau_x > n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_x(\tau_x > n) \\ &= \mathbb{E}_x \tau_x .\end{aligned}$$

Thus, if there exists x with $\mathbb{E}_x \tau_x < \infty$, then $\dfrac{\vec{\mu}^x}{\sum_{y\in\mathcal{X}} \vec{\mu}^x_y}$ defines a stationary distribution.

(ii) ⇒ (iii). Since $\sum_y \vec{\pi}_y = 1$, there exists at least one state, say x, s.t. $\vec{\pi}_x > 0$. For any fixed $y \in \mathcal{X}$, since the chain is irreducible, one has x communicates with y. That is, $\exists\, m$, s.t. $[P^m]_{xy} > 0$. Thus,

$$\vec{\pi}_y = [\vec{\pi} P^m]_y = \sum_{z \in \mathcal{X}} \vec{\pi}_z [P^m]_{zy} \geq \vec{\pi}_x \cdot [P^m]_{xy} > 0.$$

Therefore, $\vec{\pi}_y > 0$ for any $y \in \mathcal{X}$.

Thm 9.3

If I & S hold, then $\vec{\pi}_y = \dfrac{1}{\mathbb{E}_y \tau_y}$, $\forall y \in \mathcal{X}$.

Theorem 9.3 implies

$$\mathbb{E}_y \tau_y = \frac{1}{\vec{\pi}_y}, \quad \forall y \in \mathcal{X}.$$

Thus, $\mathbb{E}_y \tau_y < \infty$, $\forall y \in \mathcal{X}$.

(iii) ⇒ (i). Trivial. ∎.

Remark 15.1. Positive recurrent states are recurrent, i.e., for any $x \in \mathcal{X}$ s.t. $\mathbb{E}_x T_x < \infty$, one has $\mathbb{P}_x(T_x < \infty) = 1$.

why?

This is the end of this lecture!